

# Universal quantized spin-Hall conductance fluctuation in graphene

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We report a theoretical investigation of quantized spin-Hall conductance fluctuation of graphene devices in the diffusive regime. Two graphene models that exhibit quantized spin-Hall effect (QSHE) are analyzed. Model-I is with unitary symmetry under an external magnetic field  $B \neq 0$  but with zero spin-orbit interaction,  $t_{SO} = 0$ . Model-II is with symplectic symmetry where  $B = 0$  but  $t_{SO} \neq 0$ . Extensive numerical calculations indicate that the two models have exactly the same universal QSHE conductance fluctuation value  $0.285e/4\pi$  regardless of the symmetry. Qualitatively different from the conventional charge and spin universal conductance distributions, in the presence of edge states the spin-Hall conductance shows an one-sided log-normal distribution rather than a Gaussian distribution. Our results strongly suggest that the quantized spin-Hall conductance fluctuation belongs to a new universality class.

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One of the most important transport features of mesoscopic conductors is the *universal* conductance fluctuation (UCF) in the diffusive regime caused by disorder scattering and quantum coherence[1]. The universality characterized by the value of UCF only depends on the dimensionality and symmetry of the system. According to random matrix theory (RMT)[2], there are three ensembles or universalities due to symmetry: (1) when time-reversal and spin-rotation symmetries are present, *i.e.* when magnetic field  $B = 0$  and spin-orbit interaction (SOI)  $t_{SO} = 0$ , the Hamiltonian  $H$  of the system is an orthogonal matrix and one has circular orthogonal ensemble (COE). COE is characterized by a symmetry index  $\beta = 1$ . (2) If time-reversal symmetry is broken by  $B \neq 0$ ,  $H$  is unitary and one has the circular unitary ensemble (CUE) characterized by  $\beta = 2$ . (3) If spin-rotation symmetry is broken by  $t_{SO} \neq 0$  while time-reversal symmetry is maintained, one has the circular symplectic ensemble (CSE) for which  $\beta = 4$ . While different ensembles have different values of UCF, it is amazing that the multitudes possibilities of electron dynamics in nature can be classified by only a few ensembles[3]. For instance, in one dimension (1D) the UCF value is given by[2]  $[rms(G)]^2 = 2/(15\beta)$ .

Recently, *universal* fluctuation was also found to occur in 2D mesoscopic spin-Hall effect (SHE)[7]. SHE can be induced by spin-orbit interaction, for instance Rashba SOI in 2D, such that chemical potentials of the spin-up or -down channels become different at the two boundaries of a mesoscopic sample[5, 6]. With disorder, numerical calculations showed[7] that the spin-Hall conductance  $G_{sH}$  of a 2D mesoscopic system fluctuates from sample to sample with a value  $rms(G_{sH}) \approx 0.18e/4\pi$ : this is independent of system details thus universal, and the phenomenon is termed universal spin-Hall conductance fluctuation (USCF). The numerical value of USCF has

been quantitatively confirmed by RMT[8]. For most situations,  $G_{sH}$  itself may have any value in units of  $e/4\pi$  depending on system details. On the other hand, several authors have advanced the notion of *quantized* SHE (QSHE) for situations where electronic edge states exist: in QSHE  $G_{sH}$  takes integer multiples of  $e/4\pi$ . In particular, QSHE is shown to occur in 2D graphene due to SOI plus the peculiarity of graphene electronic structure[9]. QSHE is also predicted to occur in graphene without SOI but with an external magnetic field[10]. Therefore, using the language of RMT[2], QSHE occurs in graphene with CUE where  $B \neq 0$  but  $t_{SO} = 0$ ; and with CSE where  $B = 0$  but  $t_{SO} \neq 0$ .

Several important and interesting questions therefore arise concerning the universality of QSHE: is it still classifiable by the RMT ensembles? As the disorder is increased, is there a USCF for QSHE and if there is, is the value different from the USCF for SHE that is  $0.18e/4\pi$ ? What is the distribution of  $G_{sH}$  in QSHE? Indeed, all these questions are related to the curiosity, *i.e.* whether or not the Dirac dispersion relation of graphene brings new physics to the spin-Hall conductance fluctuation in the quantized SHE. It is the purpose of this work to investigate these issues.

To be more specific, we investigate the two graphene models that exhibit QSHE[9, 10] as mentioned above. In the first model, model-I[10], SOI is neglected in the graphene but a magnetic field is applied causing a Zeeman splitting. Model-I has unitary symmetry and importantly is in the quantum Hall regime where edge states are present. Due to the Zeeman splitting and graphene energy spectrum both electron-like and hole-like edge states exist near the Fermi level forming counter-circulating edge states in graphene that has been confirmed experimentally[11]. It is these counter-circulating edge states that lead to QSHE[10]. The sec-

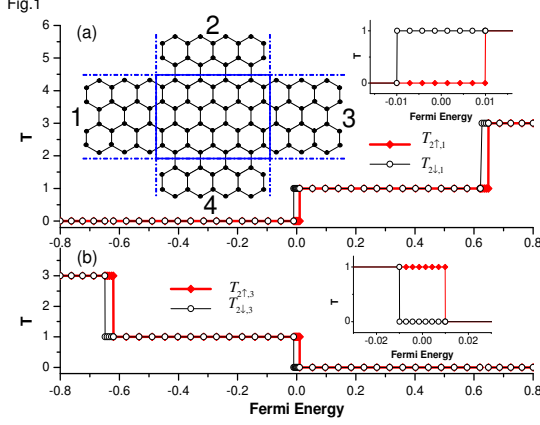


FIG. 1: (Color online) The transmission coefficient  $T_{21}$  and  $T_{23}$  versus energy at a fixed magnetic flux. Inset: schematic plot of the four terminal mesoscopic sample where the intrinsic SO interaction exists in the center scattering region and the leads 1, 3. And the Rashba SO only exists in the center part and the leads 1, 3, when the spin-Hall conductance is measured through leads 2, 4.

ond model, model-II, is the one proposed by Kane and Mele[9] where intrinsic SOI gives rise to "spin filtered" edge states that cause QSHE based on an idea discussed by Haldane[12]. Clearly, model-II has symplectic symmetry. Although the value of SOI parameter  $t_{SO}$  for graphene is small[13], model-II is nevertheless very useful for our purpose, namely to investigate universality class of QSHE. As we show later, the value of  $t_{SO}$ —as long as it is nonzero, turns out to be irrelevant as far as universality is concerned. From the symmetry point of view, one would expect these two models to belong to different universality classes. To our surprise, extensive numerical results indicate that in the presence of edge states, the QSHE dominates the physics and these two models give exactly the same universal value for  $USCF = 0.285e/4\pi$  regardless of symmetry. The distribution of  $G_{sH}$  in the QSHE regime is found to obey an one-sided log-normal distribution: this is qualitatively different from the conventional UCF for charge and USCF for SHE where it is a Gaussian distribution.

In a tight-binding representation, the Hamiltonian for 2D honeycomb lattice of graphene can be written as:

$$H_1 = \sum_{i\sigma} \epsilon_i c_{i\sigma}^\dagger c_{i\sigma} - t \sum_{\langle ij \rangle \sigma} e^{i2\pi\phi_{ij}} c_{i\sigma}^\dagger c_{j\sigma} + g_s \sum_{i\sigma} c_{i\sigma}^\dagger (\sigma \cdot \mathbf{B}) c_{i\sigma} \quad (1)$$

for model-I, and

$$H_2 = \sum_{i\sigma} \epsilon_i c_{i\sigma}^\dagger c_{i\sigma} - t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma}$$

$$+ \frac{2i}{\sqrt{3}} t_{SO} \sum_{\langle\langle ij \rangle\rangle} c_i^\dagger \sigma \cdot (\mathbf{d}_{kj} \times \mathbf{d}_{ik}) c_j \quad (2)$$

for model-II, where  $c_{i\sigma}^\dagger$  ( $c_i$ ) is the creation (annihilation) operator for an electron with spin  $\sigma$  on site  $i$ . The first term in  $H_1$  and  $H_2$  is the on-site single particle energy where diagonal disorder is introduced by drawing  $\epsilon_i$  randomly from a uniform distribution in the interval  $[-W/2, W/2]$ . Here  $W$  measures strength of disorder. The second term in  $H_1$  is due to nearest neighbor hopping and the presence of a magnetic field, the last term in  $H_1$  is due to Zeeman energy. Here  $g_s = (1/2)g\mu_B$  (with  $g = 4$ ) is the Lande  $g$  factor, phase  $\phi_{ij} = \int \mathbf{A} \cdot d\mathbf{l} / \phi_0$ ,  $\phi_0 = h/e$  is the quantum of flux, and the spin-Hall conductance and its fluctuation are in unit of  $e/4\pi$ . In  $H_2$  the last term is the SOI that involves next nearest sites of indices  $i, j$  with  $k$  the common nearest neighbor of  $i$  and  $j$ , and  $\mathbf{d}_{ik}$  describes a vector pointing from  $k$  to  $i$ .

We use the four-probe device schematically shown in the inset of Fig.1 to investigate USCF in QSHE. The four probes are exact extensions from the central scattering region, *i.e.* the probes are graphene nano-ribbons. The number of sites in the scattering region is denoted as  $N = n_x \times n_y$ , where there are  $n_x = 8 \times n + 1$  sites on  $n_y = 4 \times n$  chains. We apply external bias voltages  $V_i$  with  $i = 1, 2, 3, 4$  at the four different probes as  $V_i = (v/2, 0, -v/2, 0)$ . The spin-Hall and charge Hall conductance  $G_{sH}$  and  $G_H$  can be calculated from the multi-probe Landauer-Buttiker formula[7]:

$$G_{sH} = (e/8\pi)[(T_{2\uparrow,1} - T_{2\downarrow,1}) - (T_{2\uparrow,3} - T_{2\downarrow,3})] \\ G_H = (e^2/h)[(T_{2\uparrow,1} + T_{2\downarrow,1}) - (T_{2\uparrow,3} + T_{2\downarrow,3})] \quad (3)$$

where the transmission coefficient is given by  $T_{2\sigma,1} = \text{Tr}(\Gamma_{2\sigma} G^r \Gamma_1 G^a)$  with  $G^{r,a}$  being the retarded and advanced Green functions of the central disordered region which can be evaluated numerically. The quantities  $\Gamma_{i\sigma}$  are the linewidth functions describing coupling of the probes and the scattering region and are obtained by calculating self-energies  $\Sigma^r$  of the semi-infinite leads using a transfer matrices method[14]. The spin-Hall conductance fluctuation is defined as  $\text{rms}(G_{sH}) \equiv \sqrt{\langle G_{sH}^2 \rangle - \langle G_{sH} \rangle^2}$ , where  $\langle \dots \rangle$  denotes averaging over an ensemble of samples with different configurations of the same disorder strength  $W$ . In the following, our numerical data are mainly collected on a system with  $n = 8$ , *i.e.* with  $32 \times 65$  sites in the graphene. In the rest of the paper, we fix units by setting energy  $E$ , disorder strength  $W$ , SOI coupling  $t_{SO}$  in terms of the hopping parameter  $t$ , and the magnetic field in terms of magnetic flux  $\phi$ .

We first examine model-I which has unitary symmetry. Fig.1 shows the transmission coefficient  $T_{2\sigma,1}$  and  $T_{2\sigma,3}$  as a function of energy with  $\phi = 3\sqrt{3}/128$  and without disorder, the spin index  $\sigma = \uparrow, \downarrow$ . We observe that if we neglect the Zeeman energy the quantum charge Hall conductance takes the well known result[4]  $G_H =$

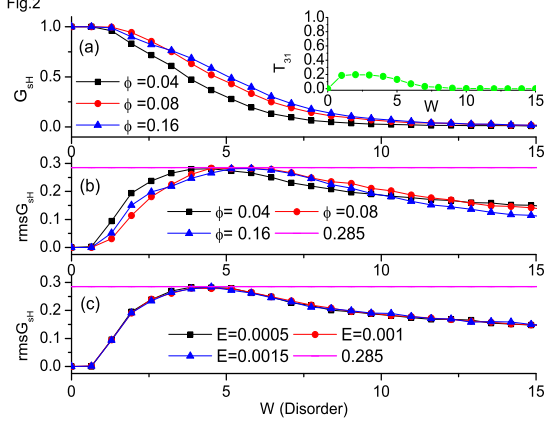


FIG. 2: (Color online) Spin-Hall conductance and its fluctuation versus disorder strength at different energies and magnetic fluxes for the first model. Inset: the transmission coefficient  $T_{31}$  versus disorders.

$\pm 4(|n| + 1/2)(e^2/h)$ . In addition, because of edge states we see that  $T_{21}$  is nonzero and  $T_{23}$  is zero above the Fermi level  $E = 0$ , while  $T_{21}$  is zero and  $T_{23}$  is nonzero below Fermi level exhibiting hole-like behavior. Due to the Zeeman shift we have  $T_{2\uparrow,1} = T_{2\downarrow,3} = 0$  and  $T_{2\downarrow,3} = T_{2\uparrow,1} \neq 0$  near Fermi level. From Eq.(3) we obtain QSHE:  $G_{sH} = 1$  in unit of  $e/4\pi$  and  $G_H = 0$ . Here we emphasize that the bias voltage is applied across probes 1 to 3 (see Fig1) and it causes a transverse flow of spin-current between probes 2 and 4 that leads to the QSHE.

In the regime of QSHE, we now increase disorder strength  $W$ . This causes a break down of the integer value of  $G_{sH}$  and induces sample to sample fluctuations of  $G_{sH}$ . Fig.2a plots the average  $G_{sH}$  by calculating 5000 samples for each point on the figure, Fig.2b plots the corresponding fluctuation  $\text{rms}(G_{sH})$ , as a function of  $W$ . When  $W$  is increased,  $G_{sH}$  decreases from its quantized value  $G_{sH} = 1$  and  $\text{rms}(G_{sH})$  increases. The break down of quantized  $G_{sH}$  is due to  $W$  that causes a direct transmission from probe 1 to 3 (see Fig.1), this is shown in the inset of Fig.2a where the direct transmission  $T_{31}$  is plotted against  $W$ . From  $T_{31}$  we conclude that the graphene device is in an insulating regime at small  $W$ , *i.e.* zero or very small  $T_{31}$ ; it is in a diffusive regime for intermediate  $W$  and finally reentrant to the insulating regime for large  $W$ . For a given  $E$  or  $\phi$ ,  $\text{rms}(G_{sH})$  develops a “plateau” region, *e.g.* in the range  $W = [3, 7]$  in Fig.2b. This plateau is at  $\text{rms}(G_{sH}) = 0.285$  in unit of  $e/4\pi$ . The plateau range of  $W$  depends on specific values of  $E$  or  $\phi$ , but we found  $\text{rms}(G_{sH}) = 0.285$  is always true if there is a plateau, *i.e.*, if the diffusive transport regime is established. We therefore identify  $\text{rms}(G_{sH}) = 0.285$  as a “universal” value. This USCF value is different from that of the conventional SHE situation[7, 8] where the

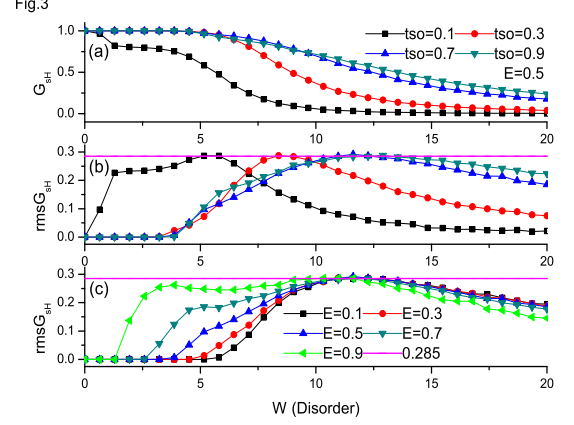


FIG. 3: (Color online) Spin-Hall conductance and its fluctuation versus disorder strength at different energies and magnetic fluxes for the second model. The parameters used in (a) and (b) are the same.

universal value is 0.18. Therefore QSHE and SHE belong to different universality classes due to this different statistical property.

Next, we investigate Model-II that has a symplectic symmetry. For such a graphene device there is an energy gap between  $-1 < E < 1$ , within which edge states exist[9]. Fig.3 plots averaged  $G_{sH}$  and  $\text{rms}(G_{sH})$  versus  $W$  for a given set of  $E$ ,  $t_{SO}$  parameter values. 5000 samples were calculated for the disorder averaging. Similar behavior is found as that of Model-I. For different values of  $t_{SO}$ ,  $\text{rms}(G_{sH})$  reaches a plateau at different range of  $W$  (see Fig.3). Amazingly, all plateaus have the same value and this value is precisely  $\text{rms}(G_{sH}) = 0.285$ ! To further confirm this finding, Fig.3c plots  $\text{rms}(G_{sH})$  vs  $W$  for a fixed  $t_{SO}$  but several different values of energy  $E$ . Again, same conclusion is obtained. This indicates that there exist a transport regime where the QSHE conductance fluctuation has a universal behavior independent of disorder (albeit a narrow region), energy and SOI. Results of Fig.2 and Fig.3 strongly suggest that there is a universal spin-Hall conductance fluctuation in the quantized spin-Hall regime with USCF= 0.285 in unit of  $e/4\pi$ . This is different from the conventional SOI induced SHE where USCF= 0.18[7].

Very importantly, it appears that symmetry does not play a role in the QSHE regime at least for the CUE and CSE cases we have examined: both give  $\text{rms}(G_{sH}) = 0.285$ . To further support this finding, we calculated the distribution function of  $G_{sH}$ ,  $P(G_{sH})$ , in the QSHE regime. Such a distribution is a Gaussian for conventional SHE in the diffusive regime[7]. For QSHE, Fig.4a-d plot  $P(G_{sH})$  for four different values of  $W$  in the universal regime for Model-II which has CSE symmetry. Data were collected by calculating 84,000 samples for each  $W$ .

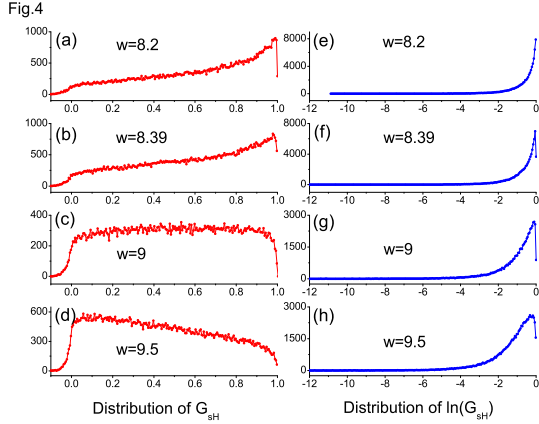


FIG. 4: (Color online)(a)-(d) The distribution of spin-Hall conductance at different disorder strengths for the second model. (e)-(f) The distribution of  $\ln(G_{sH})$ .

The distributions are completely different from a Gaussian! We found that by using  $\ln(G_{sH})$  as a variable and plot  $P(\ln(G_{sH}))$ , all the distributions become one-sided log-normal (see Fig.4e-h). For Model-I which has CUE symmetry, our results show the same conclusion, *i.e.* the distribution of quantum spin-Hall conductance is an one-sided log-normal. Therefore, for the two models we investigated, not only  $\text{USCF}_{rms}(G_{sH}) = 0.285$  is the same, but also the distribution function is the same. This strongly indicates that in the presence of edge states (*i.e.* QSHE), systems with unitary symmetry and symplectic symmetry belong to the same universality class that is different from the conventional SHE.

Finally, as a further confirmation of the QSHE universality class, we have carried out extensive calculation on spin-Hall conductance fluctuation for the same four probe graphene device with additional Rashba SOI  $t_R$ [15]. For *non-zero*  $t_R$ , three cases are of interest. (1).  $B = 0$  and  $t_{SO} = 0$ . For this situation it is obvious that there is no edge state and therefore spin Hall effect caused by  $t_R$  is not quantized. Indeed, here we did not obtain the USCF for QSHE but obtained a value of 0.18 for all energies, *i.e.*, the same as the conventional USCF found before[7, 8]. As expected, for this case the distribution of  $G_{sH}$  was found to be a Gaussian. (2). When  $|E| < 1$ , for both model I and model II our numerical results show that  $\text{USCF} = 0.285$  remains the same as long as  $t_R$  does not destroy the edge states. (3). When  $|E| > 1$ , there is no edge states in model II[9], our results show that  $\text{USCF} = 0.18$  for any  $t_{SO}$ . Therefore, edge states dominate the quantized spin-Hall physics and  $t_R$  is an irrelevant parameter (for both model I and model II). On the other hand, if edge states are absent  $t_{SO}$  becomes an irrelevant parameter (for model II). This clearly shows the landscape of universality class and it is the edge state that

drives the system from the universality of  $\text{USCF} = 0.18$  to the new universality we have discussed.

In summary, we have investigated quantized spin-Hall conductance fluctuation for two models with unitary and symplectic symmetry, respectively. Our numerical results show that both models exhibit the same universal quantum spin-Hall conductance fluctuation with the value  $0.285e/4\pi$ . Due to the presence of edge states, the distribution of quantum spin-Hall conductance obeys one-sided log-normal distribution for both models. This strongly suggests that the quantized spin-Hall conductance fluctuation for systems with both unitary symmetry and symplectic symmetry belong to the same universality class that is different from the usual spin-Hall conductance fluctuation in the absence of edge states.

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